Question: How does MWU behave in Congestion Games?

**Congestion Games**

Rosenthal, R.W. '73 \((N; E; (S_i)_{i \in N}; (c_e)_{e \in E})\) where \(N\): agents, \(E\): a set of resources with positive cost functions \(c_e\). Each player \(i\) has a strategy set \(S_i\) of subsets of \(E\) \((S_i \subseteq 2^E)\). E.g.,

\[
\Phi(s) = \sum_{e \in E} \sum_{j=1}^{c_e(j)}: \text{the potential function.}
\]

**Randomization:** Each player \(i\) has a probability distribution over \(S_i\).
- \(p_i\): probability \(i\) chooses strategy \(\gamma\) and \(c_{i\gamma}\): the expected cost of player \(i\) given that he chooses strategy \(\gamma\).

**MWU:** Exponential variant (Hedge)

The update rule (function) for MWU:

\[
p_{i\gamma}(t + 1) = p_{i\gamma}(t) \frac{(1 - \epsilon) c_{i\gamma}(t)}{\sum_{\gamma' \in S_i} p_{i\gamma'}(t)(1 - \epsilon) c_{i\gamma'}(t)},
\]

\(\forall i \in N, \forall \gamma \in S_i\), where \(\epsilon_i < 1\).

**MWU:** Linear version of MWU

The update rule (function) for MWU:

\[
p_{i\gamma}(t + 1) = p_{i\gamma}(t) \frac{1 - \epsilon c_{i\gamma}(t)}{\sum_{\gamma' \in S_i} p_{i\gamma'}(t)c_{i\gamma'}(t)},
\]

\(\forall i \in N, \forall \gamma \in S_i\), where \(\epsilon_i\) is a small constant.

**Discrete Time Dynamical Systems and Chaos**

Let \(f: X \to X\) continuous on a compact set \(X \subseteq \mathbb{R}\).

**Limit Cycle:** A periodic orbit (i.e., \(\{z_1, \ldots, z_k\}\) with \(z_{i+1} = f(z_i)\) for \(1 \leq i \leq k - 1\) and \(f(z_k) = z_1\)) that some initial conditions converge to.

**Li and Yorke Chaos:** For each \(k \in \mathbb{Z}^+\), there exists a periodic point \(p \in X\) of period \(k\) and there is an uncountably infinite set \(S \subseteq X\) that is "scrambled".

**Theorem [Li & Yorke, '75]: Period three implies chaos**

Let \(J\) be an interval and let \(F: J \to J\) be continuous. Assume there is a point \(a \in J\) for which the points \(b = F(a), c = F^2(a)\) and \(d = F^3(a)\), satisfy

\[
d < a < b < c \text{ (or } d > a > b > c).\]

Then
1. For every \(k = 1, 2, \ldots\) there is a periodic point in \(J\) having period \(k\).
2. There is an uncountable "scrambled" set \(S \subseteq J\) (containing no periodic points), which satisfies the following conditions:
   - For every \(p, q \in S\) with \(p \neq q\),
     \[
     \limsup_{n \to \infty} |F^n(p) - F^n(q)| > 0 \quad \text{and} \quad \liminf_{n \to \infty} |F^n(p) - F^n(q)| = 0.
     \]
   - For every point \(p \in S\) and periodic point \(q \in J\),
     \[
     \lim_{n \to \infty} |F^n(p) - F^n(q)| > 0.
     \]

**Convergence of MWU to Nash Equilibria**

**Theorem**

Function \(\Psi \defeq \E_{s \sim p} [\Phi(s)]\) is decreasing w.r.t. time, i.e. \(\Psi(p(t + 1)) \leq \Psi(p(t))\) where equality \(\Psi(p(t + 1)) = \Psi(p(t))\) holds only at fixed points.

Assume that the fixed points of MWU\(_t\) are isolated.

**Non-Convergence of MWU:** Limit Cycle and Chaos

**Theorem**

There exist two player two strategy symmetric congestion games such that MWU\(_{\epsilon}\) exhibits Li-Yorke chaos. There also exist such games where MWU\(_{\epsilon}\) has an uncountably infinite set of initial conditions converging to a limit cycle.

**Corollary**

For any \(1 > \epsilon > 0\) and \(n\), there exists a \(n\)-player congestion game \(G(\epsilon)\) (depending on \(\epsilon\)) so that MWU\(_{\epsilon}\) dynamics exhibits Li-Yorke chaos.